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A COMPUTER FORMULATION FOR THE ANALYSIS ON CONTINUOUS NONPRISMATIC FOLDED PLATE STRUCTURES OF ARBITRARY CROSS-SECTION

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SUMMARY:

A computer solution to analyze nonprismatic folded plate structures is shown. Arbitrary cross-sections (simple and multiple), continuity over intermediate supports and general loading and longitudinal boundary conditions are dealt with. The folded plates are assumed to be straight and long (beam like structures) and some simplifications are introduced in order to reduce the computational effort. The formulation here presented may be very suitable to be used in the bridge deck analysis.

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1. INTRODUCTION

The theories of folded plate structures have been developed for a very long time. A review of some of them can be seen in [1] . Important contributions to long simple supported folded plates, using harmonic expansion and matrix techniques are [2] and [3] . Using the same techniques and Theory of Elasticity, the extension of the above results, to study short and long simple supported structures, has been introduced in Spain in the reference [4] .

The continuous folded plate structures have presented new difficulties in their analysis. Some of them have been solved by means of different techniques. Fourier expansion using Rayleigh (or Inglis) functions has been applied in [5] to study single span structures with other transversal conditions than the simply support and also extended there to structures with several spans. Flexibility methods have been succesful developed in publications [6] and [7] .

The contributions to the analysis of nonprismatic folded plate structures are more scarce. Often general computer methods of analysis as the Finite Element Method have been used [6]. However in this method, the peculiarity of this type of structures are not fully exploited. Finite strip and finite segment methods are possible alternatives.

Born ([8] and [9]) has considered the analysis of pyramidal and prismoidal folded plates. Johnson and Ti-ta Lee have developed in [10] a theoretical and experimental analysis of long nonprismatic folded plate structures. Later a computer program and a model test based in the above analysis have been developed in the Laboratorio Central. Between the results obtained there, a comparative study has been carried out [11] . However the method presented in the reference [10] has some important limitations, namely, a) Only simple supported structures are considered. b) Multiple transversal cross-sections can not deal with in the analysis, i. e. folded plate structures with more than two plates meeting at a joint. c) General loading and longitudinal boundary conditions are excluded in the formulations.

In the present paper, a natural extension of the theory developed in [10] is shown and the above limitations no longer exist. Then, structures as box girder bridges multiple cellular plates with one or several spans, can be studied by means of this extended method of structural analysis. Based in this theory, a computer program is now under development, and comparative studies with alternative structural analysis will be presented in a future publication.

2. MAIN ASSUMPTIONS

The following structural method lies in the framework of a geometrically and material linear and elastic theory. Besides, the following additional assumptions are introduced:

1) The material is homogeneous

2) The structure is monolithic

3) Every plate element has the following properties: Its greatest depth is small in comparison to its plate length. As a consequence, the longitudinal behaviour can be studied as a continuous one way slab action, i.e., no longitudinal bending and torsional moments exist.

4) For simplicity, the shear and axial (transversal direction) strains are neglected.

5) The folded plate is a right structure, i.e., there exist a straight line normal to all the support plans containing the centroids of every transversal section (parallel to the support plans) of the folded plate.

6) All the supports are planes restraining only the in-plane movements (gable conditions).

3. AN OUTLINE OF THE METHOD

This method follows similar steps as the usual folded plate analysis [10] , but now the basic unknowns will be different. The main computational steps are:

1) Divide longitudinally the folded plate structure by a number A + 1 of equally spaced transversal sections (Fig. 1)

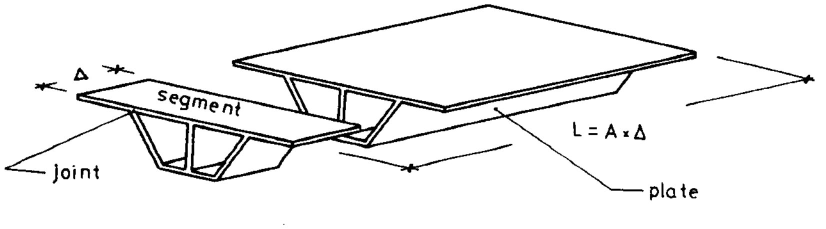

Fig. 1

2) Each segment of folded plate between two consecutive transversal sections is assumed to have its joints fixed in the transversal direction, i.e., no horizontally and vertically displacements occur (Fig. 2). That can be obtained by introducing a set of ficticious temporary supports (or reactions).(*)

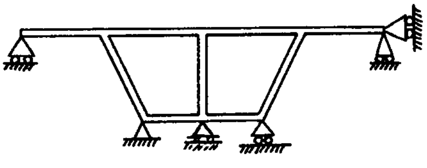

Fig. 2

3) A transversal bending analysis is carried out for each segment. Only the loading applied directly on the segment is considered in this analysis (Fig. 3). Matrix stiffness methods are used to obtain the joint rotations, stress - resultants at plate edges - bending moments, shear stresses and axial forces - and unknowns reactions at the actual and ficticious supports, called respectively:

$\theta^{(1)}_{na}$, $\mu^{(1)}_{jia}$, $\gamma^{(1)}_{jia}$, $\nu^{(1)}_{jia}$ and $R^{(1)}_{na}$ where $j = 1, 2$

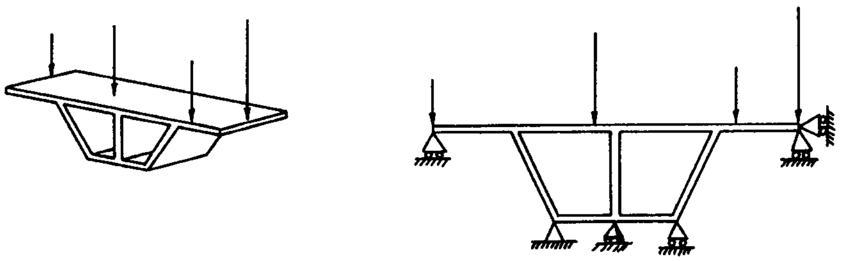

Fig. 3

(*) A set of supports needed to restrain the movement on the transversal section (without extensional deformation) can be automatically obtained, as it will be seen later.

4) In [10] the opposite reactions at the ficticious supports are equilibrated by the in-plane forces of the plates meeting at the supported joint, namely p_{1ia} or p_{2ia} (Fig. 4).

These forces can be computed directly only if the folded plate structure presents a non-multiple cross-section as has been assumed in the reference [10] . However, in the general case, a different approach has to be followed. The in-plane forces at every plate and segment ($p_{ia} = p_{1ia} + p_{2ia}$) are considered as basic unknowns through all the analysis (Fig. 4).

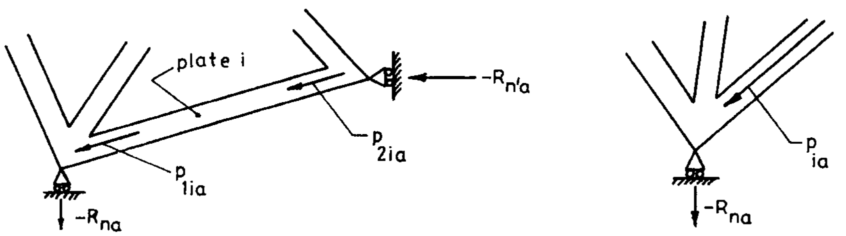

Fig. 4

5) A longitudinal analysis is carried out for each plate i. The assumed applied loading are the basic unknowns (p_{ia}) and the actual longitudinal loading (prestress actions, temperature, longitudinal forces, etc.) acting along the plate. If the folded plate is a continuous structure , i.e., with intermediate transversal supports, or is supported in different way to simply support (gable conditions), then two more unknowns must be introduced at the ends of every span of the plate i, namely G_{1i} and G_{2i} (Fig. 5).

The results of this analysis are the following stress-resultants, at each transversal section and plate: longitudinal axial forces, longitudinal bending moments and longitudinal shear forces, called respectively:

NL_{ia} , ML_{ia} and QL_{ia}

They are computed in terms of the known applied loads and the above basic unknowns.

6) The longitudinal compatibility along the joints is set up. That means, in-plane shear stresses (q_{1ia} and q_{2ia}) distributed along each plate edge must be introduced in the analy-

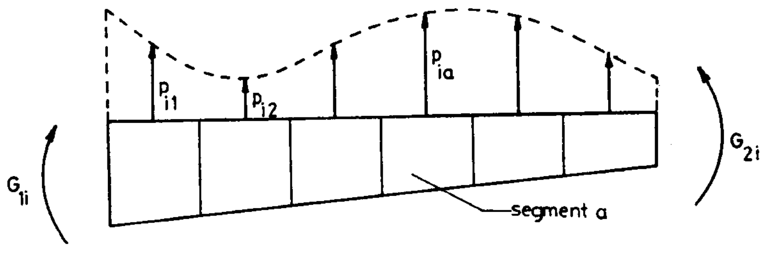

Fig. 5

sis. These stresses can be computed in terms of the basic unknowns (p_{ia} , G_{1i} and G_{2i}) by using compatibility and equilibrium conditions at each section and joint (Fig. 6).

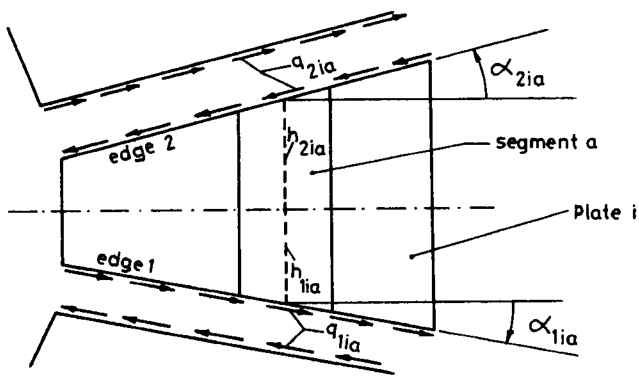

Fig. 6

7) In this step, the longitudinal analysis already done in the step 5) is again carried out taking into account the in-plane shear stresses (q_{1ia} and q_{2ia}). These stresses modify the results of the previous longitudinal analysis. The total stress-resultants of this analysis will be called:

$\overline{NL}_{ia}$, $\overline{ML}_{ia}$ and $\overline{QL}_{ia}$.

8) From the final results of the longitudinal analysis the values of the in-plane deflections (u_{ia}) at each cross-section and the end rotations (Θ_{1i} and Θ_{2i}), for each plate i, can be obtained from elementary beam theory.

9) At each joint and cross-section the transversal compatibility must be imposed. That means, plates meeting at a joint must rotate in order to an unique joint displacement occurs. Because of the structural monolithism , these plate rotations introduced transversal stress-resultants that can be computed in a similar way as in the step 3 (Fig. 7). The results of this analysis will be denoted by the superscript (2), i.e.,

$\Theta_{na}^{(2)}$, $\mu_{1ia}^{(2)}$, $\mu_{2ia}^{(2)}$, $\gamma_{1ia}^{(2)}$, $\gamma_{2ia}^{(2)}$, $\nu_{1ia}^{(2)}$, $\nu_{2ia}^{(2)}$ and $R_{na}^{(2)}$.

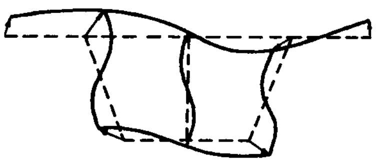

Fig. 7

10) In this step the basic unknowns (p_{ia}) and (G_{1i} and G_{2i}) are computed by means of the following conditions: a) At each cross-section and ficticious supported joint, the condition $R_{na}^{(1)} + R_{na}^{(2)} = 0$ holds, i.e., no actual support reaction exists. b) At each transversal intermediate support and each end support with different conditions to the gable ones, the continuity must be enforced (compatibility conditions). This can be acomplished by equalling the end rotations (Θ_{1i} or Θ_{2i}) between two consecutive plates.

11) Once the values p_{ia} , G_{1i} and G_{2i} are known, all the main results of interest in the analysis can be computed by back substitution.

4. DEFINITIONS

A trirectangular cartesian counterclockwise coordinate system (X, Y, Z) is introduced. The horizontal axis Z is normal to the support planes. The horizontal axis X is normal to Z and the axis Y is vertical upward. They are called global or general system (Fig. 8-a).

For every plate, two longitudinal edges can be considered and called 1 and 2 respectively. Then, the local cartesian coordinate system (x' , y' , z') can be introduced for each plate, where axis z' coincides with the general axis **Z**, and axis x' is directed from extreme 1 towards 2. The axis y' is orthogonal to x' and z' (Fig. 8-b).

The position of each plate is defined at each croos-section by the relative coordinates (h_{ia} and v_{ia}) of the extreme (edge) 2 respect to 1 (Fig. 8-c).

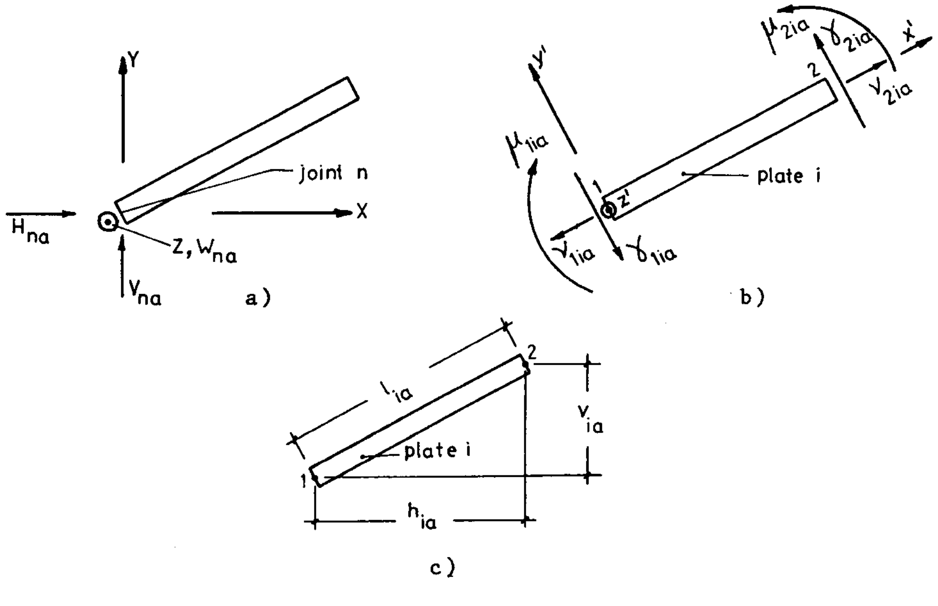

Fig. 8

Along each joint four degrees of freedom (dof) are considered. They are defined in terms of the global axis (Fig. 9), except the longitudinal displacement w_{na} that coincides with the joint direction.

General imposed boundary conditions at each joint are considered in the computation. The stress-resultants of each plate are refered to local axis and their positive values are represented in Fig. 8-b.

The external loads acting along joints are defined in global axis (fig. 8-a) and the loading on each plate are represented in the local axis of the plate.

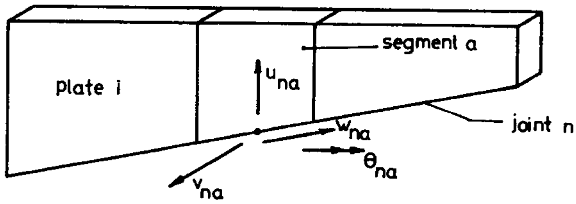

Fig. 9

5. MAIN COMPUTATIONAL STEPS OF THE ANALYSIS

In the following the main formulae used in the different steps of the analysis, described in general terms in chapter 3 are now summarized.

5.1 First transversal analysis

Each segment "a" can be analyzed as a plane frame structure with its geometric properties corresponding to its central cross-section. The only active dof are the joint rotations $\theta_{na}^{(1)}$, because sway has been assumed to be restrained by ficticious supports. The values of these rotations can be obtained from the following matrix equation:

$$\underline{K}_a \cdot \underline{\theta}_a^{(1)} = \underline{M}_a + \underline{\bar{M}}_a^{(1)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where

$\underline{K}_a$ is the stiffness matrix of the plane frame structure obtained by standard matricial analysis.

$\underline{\theta}_a^{(1)} = \{\theta_{na}^{(1)}\}$; $\underline{M}_a = \{M_{na}\}$; $\underline{\bar{M}}_a = \{\bar{M}_{na}^{(1)}\}$ and $n = (1, N)$

M_{na} is the external moment acting directly at joint "n" and

$\bar{M}^{(1)}_{na}$ is obtained by standard techniques from the equivalent end moments of the external applied loads at each plate (fixed-end moments).

Solving the equation (1), the values $\Theta^{(1)}_{na}$ are obtained. Then, the distribution on the cross-section of the transversal stress-resultants is known. Particularly their values at the extremes 1 and 2 of a plate "i" are computed from the following formulae:

-Bending moments:

$$\mu^{(1)}_{1ia} = -(r_{1ia}.\Theta^{(1)}_{1ia} + g_{2ia}.r_{2ia}.\Theta^{(1)}_{2ia}) + \bar{\mu}^{(1)}_{1ia} = \bar{\bar{\mu}}^{(1)}_{1ia} + \bar{\mu}^{(1)}_{1ia}$$

$$\mu^{(1)}_{2ia} = (r_{2ia}.\Theta^{(1)}_{2ia} + g_{1ia}.r_{1ia}.\Theta^{(1)}_{1ia}) + \bar{\mu}^{(1)}_{2ia} = \bar{\bar{\mu}}^{(1)}_{2ia} + \bar{\mu}^{(1)}_{2ia}$$

where r_{1ia} and r_{2ia} are the stiffness coefficients and g_{1ia} and g_{2ia} are the carry-over factors.

$\Theta^{(1)}_{1ia}$ and $\Theta^{(1)}_{2ia}$ are the values of the rotations at each transversal extreme of the plate, equal to the corresponding joint rotation $\Theta^{(1)}_{na}$.

$\bar{\mu}^{(1)}_{1ia}$ and $\bar{\mu}^{(1)}_{2ia}$ are the bending moments corresponding to the initial solution (no rotations at the joints).

-Shear forces:

$$\gamma^{(1)}_{jia} = -\frac{\bar{\bar{\mu}}^{(1)}_{2ia} - \bar{\bar{\mu}}^{(1)}_{1ia}}{l_{ia}} + \bar{\gamma}_{jia} \qquad \text{and } j = 1,\ 2$$

-Axial forces:

$$\nu^{(1)}_{1ia} = \nu_{1ia} + \bar{\nu}_{1ia} \ ; \quad \nu^{(1)}_{2ia} = \nu_{2ia} + \bar{\nu}_{2ia}$$

They are obtained by setting up two equilibrium equations at each joint "n" and one equilibrium equation for each plate "i":

-Horizontal forces at "n":

$$\sum_{i\varepsilon N_1} \nu^{(1)}_{1ia}\frac{h_{ia}}{l_{ia}} - \sum_{i\varepsilon N_2} \nu^{(1)}_{2ia}\frac{h_{ia}}{l_{ia}} + \delta_{nn'}.R^{(1)}_{na} = -H_{na} +$$

$$+ \sum_{i\varepsilon N_1} \gamma^{(1)}_{1ia}\frac{v_{ia}}{l_{ia}} - \sum_{i\varepsilon N_2} \gamma^{(1)}_{2ia}\frac{v_{ia}}{l_{ia}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2\text{-a})$$

-Vertical forces at "n":

$$\sum_{i\varepsilon N_1} \nu^{(1)}_{1ia} \frac{v_{ia}}{l_{ia}} - \sum_{i\varepsilon N_2} \nu^{(1)}_{2ia} \frac{v_{ia}}{l_{ia}} + \delta_{nn'} R^{(1)}_{na} = -V_{na} -$$

$$- \sum_{i\varepsilon N_1} \gamma^{(1)}_{1ia} \frac{h_{ia}}{l_{ia}} + \sum_{i\varepsilon N_2} \gamma^{(1)}_{2ia} \frac{h_{ia}}{l_{ia}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2\text{-b})$$

-Horizontal forces at "i":

$$\nu_{1ia} + \bar{\nu}_{1ia} + \nu_{2ia} + \bar{\nu}_{2ia} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2\text{-c})$$

where

ν_{1ia} and ν_{2ia} are the unknown axial forces at transversal extremes 1 and 2 of the plate "i".

H_{na} and V_{na} are the external forces applied to the joint "n".

$\begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix}$ are the set of plates "i" wich extremes $\begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$ coincides with the joint "n".

$\delta_{nn'}$ is the Kronecker delta, i.e. $\delta_{nn'} = 0$ if $n \neq n'$ and $= 1$ if $n = n'$.

Summarizing all the equations (2), for each joint "n", the following matrix equation can be obtained

$$\underline{B} \cdot \underline{S}^{(1)} = \underline{P}^{(1)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

where $\underline{B}$ is a non singular square matrix provides the proper ficticious supports have been introduced. All elements of this matrix are known, depending only on the geometric properties and support conditions of the cross-section.

$\underline{S}^{(1)}$ = unknowns vector = $\{\underline{\nu}_{1a}, \underline{\nu}_{2a}, \underline{R}^{(1)r}_a, \underline{R}^{(1)f}_a\} = \{\underline{S}_1, \underline{R}^{(1)f}_a\}$

$\underline{\nu}_{1a} = \{\nu_{1ia}\}$; $\underline{\nu}_{2a} = \{\nu_{2ia}\}$ where $i = (1, I)$

$\underline{R}^{(1)r}_a = \{R^{(1)r}_{na}\}$ and $n = (1, N_r)$

$\underline{R}^{(1)f}_a = \{R^{(1)f}_{na}\}$ and $n = (1, N_f)$

$R^{(1)r}_{na}$ is the n-th actual existing support reaction at joint

$R^{(1)f}_{na}$ is the n-th ficticious imposed support reaction at joint

$\underline{P}^{(1)}$ is the known vector collecting all the external forces

Besides the equation (3), there exist the following conditions, expresing the fact of non-existence of the ficticious supports, i.e.,

$$\underline{R}^{(1)f} = \underline{0} \quad \ldots\ldots(4)$$

Partitioning the equation (3) in the following way(*)

$$\begin{bmatrix} \underline{B}_{11} & \underline{B}_{12} \\ \underline{B}_{21} & \underline{B}_{22} \end{bmatrix} \cdot \begin{Bmatrix} \underline{S}_1 \\ \underline{R}_a^{(1)f} \end{Bmatrix} = \begin{Bmatrix} \underline{P}_1^{(1)} \\ \underline{P}_f^{(1)} \end{Bmatrix} \quad \ldots\ldots(5)$$

and taking into account equation (4), the following conditions should be fulfilled:

$$\underline{B}_{11} \cdot \underline{S}_1 = \underline{P}_1^{(1)} \qquad \text{and} \qquad \underline{B}_{21} \cdot \underline{S}_1 = \underline{P}_f^{(1)}$$

or equivalently

$$\underline{B}_{21} \cdot \underline{B}_{11}^{-1} \cdot \underline{P}_1^{(1)} = \underline{P}_f^{(1)} \quad \ldots\ldots(6)$$

Generally, the condition (6) is not satisfied and therefore the distorsion on the cross-section of the folded plate must be considered in the analysis.

5.2 Longitudinal bending analysis

Each plate is considered here as a simply supported beam loaded with a set of unknown in-plane forces p_{ia} (a = 1, 2,..., A) and two end bending moments G_{1i} and G_{2i}.

The longitudinal stress-resultants are computed from the elementary beam theory (Fig. 5):

$$\underline{ML}_i = \underline{ML}_{0i} + \underline{S}_9 \cdot \underline{p}_i + G_{1i}\,(\underline{e} - \underline{x}) + G_{2i} \cdot \underline{x} \quad \ldots\ldots(7\text{-a})$$

$$\underline{QL}_i = \underline{QL}_{0i} + \underline{S}_{10} \cdot \underline{p}_i + \frac{1}{L}(G_{1i} - G_{2i}) \cdot \underline{e} \quad \ldots\ldots(7\text{-b})$$

$$\underline{NL}_i = \underline{NL}_{0i} \quad \ldots\ldots(7\text{-c})$$

where $\underline{ML}_i = \{ML_{ia}\}$; $\underline{QL}_i = \{QL_{ia}\}$; $\underline{NL}_i = \{NL_{ia}\}$

$$\underline{ML}_{0i} = \{ML_{0ia}\};\ \underline{NL}_{0i} = \{NL_{0ia}\};\ \underline{QL}_{0i} = \{QL_{0ia}\};\ \underline{p}_i = \{p_{ia}\}$$

(*) These ficticious supports are used only for convenience. Mathematically $\underline{B}_{11}$ can be obtained as a principal non singular submatrix of $\underline{B}$, which order is the rank of $\underline{B}$.

and $\underline{x} = \{(2a-1)\Delta/2L\}$ where $a = (1, A)$

$\underline{e} = \{1\}$; $\underline{S}_9 = L(\underline{e} - \underline{x})\underline{S}_5 + \underline{S}_7(\underline{S}_1+\underline{S}_2) + \underline{S}_8(\underline{S}_3+\underline{S}_4) + \underline{S}_3$

$\underline{S}_{10} = \underline{e}\cdot\underline{S}_5 + \underline{S}_8(\underline{S}_1+\underline{S}_2) + \underline{S}_1$; $\underline{S}_6 = -[\ \underline{e}^T(\underline{S}_1+\underline{S}_2) + \underline{S}_5]$

and $\underline{S}_5 = -[\ \underline{x}^T(\underline{S}_1+\underline{S}_2) + \frac{1}{L}\ \underline{e}^T(\underline{S}_3+\underline{S}_4)]$

The subscript (0) refers to the prestress.
The signification of the other matrices will be shown later.

The formulae (7) have been obtained without considering the longitudinal displacement compatibility along the joint where adjacent plates meet. That implies the existence of the matching shear stresses q_{1ia} and q_{2ia}.

Then the longitudinal stress-resultants are computed now according to the following expresions:

$$\underline{\overline{ML}}_i = \underline{ML}_i + \underline{ML}_i^* + \underline{D}_{1i} \cdot \underline{T}_{1i} + \underline{D}_{2i} \cdot \underline{T}_{2i} \quad \text{..............(8-a)}$$

$$\underline{\overline{NL}}_i = \underline{NL}_i + \underline{NL}_i^* + \underline{D}_{3i} \cdot \underline{T}_{1i} - \underline{D}_{4i} \cdot \underline{T}_{2i} \quad \text{..............(8-b)}$$

$$\underline{\overline{QL}}_i = \underline{QL}_i + \underline{QL}_i^* - \underline{D}_{5i} \cdot \underline{T}_{1i} - \underline{D}_{6i} \cdot \underline{T}_{2i} \quad \text{..............(8-c)}$$

where

$$\underline{T}_{1i} = \{T_{1ia}\} \ ; \ \underline{T}_{2i} = \{T_{2ia}\} \quad \text{and} \quad T_{jia} = \frac{dq_{jia}}{dx} \quad (j=1,2)$$

The values of the unknowns $\underline{T}_{1i}$ and $\underline{T}_{2i}$ are computed by imposing the equilibrium and compatibility at each joint and the boundary conditions at the longitudinal ends of each plate (Fig. 10), i.e.

Equilibrium: $\sum_{i\varepsilon N_1} q_{1ia} - \sum_{i\varepsilon N_2} q_{2ia} = L_{na}$ ....(9-a)

Compatibility (free longitudinally joint):

$$\frac{1}{E}\ f_{pia} = \frac{1}{E}\ f_{qja} \quad \text{......................(9-b)}$$

where: $i\varepsilon N_p$; $j\varepsilon N_q$ and $p, q = 1$ or 2

Compatibility (non-free longitudinally joint):

$$\frac{1}{E}\ f_{pia} = 0 \quad \text{............................(9-c)}$$

where: $i \in N_p$ and $p = 1$ or 2

Boundary conditions at longitudinal ends:

The values of T_{ji0} are known(*) and defined by T^*_{ji0} $(j=1,2)$

The longitudinal bending stresses f_{mia} can be computed from the bending theory, i.e.,

$$f_{mia} = \frac{1}{Z_{mia}} (ML_{ia} + \frac{1}{2} h_{1ia} . \cos \alpha_{1ia} (T_{1ia} + T_{1ia-1}) + \frac{1}{2} h_{2ia} . \cos \alpha_{2ia} (T_{2ia} + T_{2ia-1})) \frac{1}{\cos^2 \alpha_{2ia}} + \frac{1}{A_{ia}} (NL_{ia} + \frac{1}{2} \cos \alpha_{1ia} . (T_{1ia} + T_{1ia-1}) - \frac{1}{2} \cos \alpha_{2ia} . (T_{2ia} + T_{2ia-1})) \frac{1}{\cos^2 \alpha_{2ia}} +$$

$$+ \frac{(-1)^{m+1}}{\Delta t_{mia}} . 2 \tan \alpha_{mia} . (T_{mia} - T_{mia-1}) \quad \text{..............} (10)$$

where $m = 1, 2$

A_{ia} is the cross-sectional area of plate "i" at segment "a"

Z_{mia} is the section modulus at transversal edge "m"

t_{mia} is the thickness of the plate at edge "m"

h_{mia} and α_{mia} are shown in Fig. 6

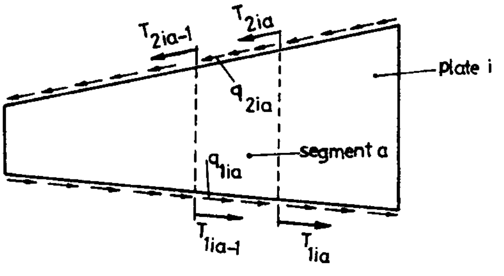

Fig. 10

(*)At the first gable support for the first span $T^*_{ji0} = 0$, and for a current span T^*_{ji0} is iqual to the value of T_{jiA} of the previons span.

Collecting all the equations (9), the following matrix equation can be deduced

$$\underline{AA} \cdot \underline{T} = \underline{BB} \quad \dots\dots \quad (11)$$

where:

$\underline{T} = \{\underline{T}_i\}$ and $\underline{T}_i = \begin{Bmatrix} \underline{T}_{1i} \\ \underline{T}_{2i} \end{Bmatrix}$

This equation already obtained in reference [10] for simple transversal sections is the wellknown Five Shear Equation. Its solution can be expressed as follows:

$$\underline{T} = \underline{AA}^{-1} \cdot \underline{BB} \quad \dots\dots \quad (12)$$

i.e.

$$\underline{T}_i = \underline{C}_{0i} + \underline{C}_{1i} \cdot \underline{G}_1 + \underline{C}_{2i} \cdot \underline{G}_2 + \underline{C}_{3i} \cdot \underline{P} \quad \dots\dots \quad (12\text{-a})$$

where:

$\underline{G}_1 = \{G_{1i}\}$; $\underline{G}_2 = \{G_{2i}\}$; $\underline{p} = \{\underline{p}_i\}$

Introducing equation (12) into the expresion (8), the following results are reached:

$$\underline{\overline{ML}}_i = \underline{\overline{ML}}_{0i} + \underline{\overline{ML}}_{1i} \cdot \underline{G}_1 + \underline{\overline{ML}}_{2i} \cdot \underline{G}_2 + \underline{\overline{ML}}_{3i} \cdot \underline{p} \quad \dots\dots \quad (13\text{-a})$$

$$\underline{\overline{NL}}_i = \underline{\overline{NL}}_{0i} + \underline{\overline{NL}}_{1i} \cdot \underline{G}_1 + \underline{\overline{NL}}_{2i} \cdot \underline{G}_2 + \underline{\overline{NL}}_{3i} \cdot \underline{p} \quad \dots\dots \quad (13\text{-b})$$

$$\underline{\overline{QL}}_i = \underline{\overline{QL}}_{0i} + \underline{\overline{QL}}_{1i} \cdot \underline{G}_1 + \underline{\overline{QL}}_{2i} \cdot \underline{G}_2 + \underline{\overline{QL}}_{3i} \cdot \underline{p} \quad \dots\dots \quad (13\text{-c})$$

From the above values, the in-plane deflections $\underline{u}_i$ along the plate "i" and its end rotations Θ_{1i} and Θ_{2i} can be obtained by elementary beam theory, i.e.,

$$\Theta_{1i} = \underline{S}_{c5} \cdot \underline{p}_{ci} \quad \dots\dots \quad (14\text{-a})$$

$$\Theta_{2i} = \underline{S}_{c6} \cdot \underline{p}_{ci} \quad \dots\dots \quad (14\text{-b})$$

$$\underline{u}_i = \underline{S}_{c9} \cdot \underline{p}_{ci} \quad \dots\dots \quad (15)$$

where:

$\underline{p}_{ci} = \{p_{cia}\}$; $\underline{u}_i = \{u_{ia}\}$; $p_{cia} = \overline{ML}_{ia}/E.I_{ia}$

I_{ia} = moment of inertia of the plate "i" at segment "a"

E = modulus of elasticity of the material

$$\underline{S}_{c5} = -(\underline{x}^T(\underline{S}_{c1}+\underline{S}_{c2})+\frac{1}{L}\ \underline{e}^T(\underline{S}_{c3}+\underline{S}_{c4}))$$

$$\underline{S}_{c6} = -(\underline{e}^T(\underline{S}_{c1}+\underline{S}_{c2})+\ \underline{S}_{c5})$$

$$\underline{S}_{c9} = L.(\underline{e} - \underline{x}).\ \underline{S}_{c5} + \underline{S}_7(\underline{S}_{c1}+\underline{S}_{c2})+\ \underline{S}_8(\underline{S}_{c3}+\underline{S}_{c4})+\underline{S}_{c3}$$

The other matrices will be specified later.

The equation (15) can be written, taking into account (13-a) in the following way

$$\underline{u}_i = \underline{u}_{0i} + \underline{u}_{1i}.\underline{G}_1 + \underline{u}_{2i}.\underline{G}_2 + \underline{u}_{3i}.\underline{P} \quad \ldots\ldots\ldots\ldots\ldots\ldots(15\text{-a})$$

5.3 Transversal compatibility. Second transversal analysis

Due to the structural monolithism, at each joint "n" an unique displacement occurs. That implies plates meeting at joint "n" should have the same total displacement. Therefore the plates must rotate in order to fullfill this condition, taking into account the values of the in-plane deflections $\underline{u}_i$ given by equation (15-a). The rotation of the plate "i" is given by the following expresion:

$$\omega_{ia} = \frac{w_{2ia} - w_{1ia}}{l_{ia}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

where w_{kia} can be computed from the joint compatibility equations given below, according to the particular joint conditions:

a) Free joint(*):

$$w_{kia} = -\frac{\cos\phi_{ija}}{\sin\phi_{ija}}\ .\ u_{ia} + \frac{u_{ja}}{\sin\phi_{ija}} \quad \ldots\ldots\ldots\ldots\ldots (17\text{-a})$$

where "i" and "j" are plates meeting at the same joint "n" $\phi_{ija} = \phi_{ja} - \phi_{ia}$ and ϕ_{ia} is the transversal angle at segment "a" between the plate "i" and the horizontal, positive counterclockwise.

b) Restrained horizontally joint displacement:

(*) If there are only two paralell plates meeting at the joint the value of w_{kia} remains unknown, but an extra condition can be imposed, namely $u_{ia} = u_{ja}$.

$w_{kia} = u_{ia} \cdot h_{ia} / v_{ia}$ .... (17-b)

and $v_{ia} \neq 0$

c) Restrained vertically joint displacement:

$w_{kia} = - u_{ia} \cdot v_{ia} / h_{ia}$ .... (17-c)

and $h_{ia} \neq 0$

d) Total restrained joint displacement:

$u_{ia} = 0$ .... (17-c)

Introducing into the equation (16) the values given by (17) and the expresion (15-a), the final formula can be stated:

$\omega_{ia} = \omega_{0ia} + \omega_{1i} \cdot \underline{G}_1 + \omega_{2i} \cdot \underline{G}_2 + \omega_{3i} \cdot \underline{P}$ .... (18)

Considering the transversal section of the folded plate as a plane frame structure, the above rotations produce the following stress-resultants:

$$\mu^{(2)}_{kia} = \bar{\mu}^{(2)}_{kia} + (-1)^k \cdot (r_{kia} \cdot \Theta^{(2)}_{kia} + g_{jia} \cdot r_{jia} \cdot \Theta^{(2)}_{jia})$$

$$\gamma^{(2)}_{kia} = \frac{\mu^{(2)}_{1ia} - \mu^{(2)}_{2ia}}{l_{ia}}$$

where k, j = 1, 2   and $j \neq k$

$\nu^{(2)}_{kia}$ are computed together with the reactions $R^{(2)}_{na}$ by the equation $\underline{B} \cdot \underline{S}^{(2)} = \underline{P}^{(2)}$ wich is obtained in similar way as the expresion (3).

$\Theta^{(2)}_{kia}$ are the values of the rotations at each transversal extreme of the plate, equal to the corresponding joint rotation.

$\Theta^{(2)}_{na}$ wich is obtained from stiffness matrix equation

$$\underline{K}_a \cdot \Theta^{(2)}_a = \underline{\bar{M}}^{(2)}_a$$

$\bar{\mu}^{(2)}_{kia}$ are the fixed end-moments at the extreme "k" of plate "i"

$$\bar{\mu}^{(2)}_{kia} = (-1)^{k+1} \cdot (r_{kia} + g_{jia} \cdot r_{jia}) \cdot \omega_{ia}$$

$\underline{\bar{M}}^{(2)}_a$ is deduced from the fixed end-moments at each plate in a similar way as has been done before for $\underline{\bar{M}}_a^{(1)}$.

5.4 Final simultaneous equations

The final values of the transversal stress-resultants are simply obtained from summation of the two previous transversal analysis, i.e.,

$$\mu_{kia} = \mu_{kia}^{(1)} + \mu_{kia}^{(2)} ; \qquad \gamma_{kia} = \gamma_{kia}^{(1)} + \gamma_{kia}^{(2)} \quad \text{and}$$

$$\nu_{kia} = \nu_{kia}^{(1)} + \nu_{kia}^{(2)}$$

Similarly, the final reactions are $R_{na} = R_{na}^{(1)} + R_{na}^{(2)}$

The values of the unknowns p_{ia} and G_{1i} and G_{2i} should be obtained such that the values of the ficticious reactions are null. That implies, the fact that the condition (6) is satisfied for the final values of $\underline{P}_1$ and $\underline{P}_f$, i.e.,

$$\underline{B}_{21}\cdot\underline{B}_{11}^{-1}\cdot\underline{P}_1 = \underline{P}_f \quad \dots\dots\dots\dots\dots\dots \quad (19)$$

where

$\underline{P}_1 = \underline{P}_1^{(1)} + \underline{P}_1^{(2)}$ and $\underline{P}_f = \underline{P}_f^{(1)} + \underline{P}_f^{(2)}$

where $\underline{P}_1^{(2)}$ and $\underline{P}_f^{(2)}$ are computed in similar way as it has been done for $\underline{P}_1^{(1)}$ and $\underline{P}_f^{(1)}$, i.e., using expresions (2-a) and (2-b) setting $H_{na} = V_{na} = 0$ and changing superscript (1) by (2).

Besides the above equilibrium equations, the compatibility conditions between two consecutive plates at the ends of each of them (equal rotations Θ_1^i and Θ_2^i at each transversal support) must be introduced by using the expresions (14-a) and (14-b).

The number of equations coincides with the number of unknowns and the solution of the problem is possible. For example, for a continuous two span folded plate structure simply supported at ends, the following results holds:

Number of unknowns: p_{ia} ---------------------- 2.I.A.

G_{1i} and G_{2i} --------------- $I^{(*)}$

Number of equations:

Transversal compatibility (17) ------------- $2(2I-N+N_r)A$

Equilibrium (19) --------------------------- $2(N-I-N_r)A$

Longitudinal compatibility ($\Theta_{ki} = \Theta_{k'j}$) ---- $I^{(*)}$

Once obtained the values of the basic unknowns, the remain results of the analysis can be computed by back-substitution on the main formulae.

6. CONCLUSIONS

A general method of elastic linear analysis on folded plate structures is shown. The procedure used represents a natural extension to the previous one given in reference [10]. The main features of the method are concerned to the general treatment of continuous folded plate structures, multiple (cellular, etc.) cross-sections and general applied loading.

In order to obtain this generality in the analysis a set of basic unknowns (in-plane forces along each plate and longitudinal bending moments at each support and plate) has been used.

The method here presented seems to be adecuate to analyze efficiently this type of structures without use more general and powerful procedures such as finite element, finite segment or finite strip methods, but they usually are more expensive in computer and man time.

7. NOTATIONS AND ABBREVIATIONS

L	=	Span length
A	=	Number of segments
I	=	Number of plates on the cross-section
N	=	Number of joints on the cross-section
X, Y, Z	=	General co-ordinate system
x', y', z'	=	Local co-ordinate system
h_{ia} , v_{ia}	=	Relative co-ordinate system
l_{ia}	=	Width of plate "i" at segment "a"
$\underline{K}_a$	=	Stiffness matrix of the plane frame structure corresponding to the central cross-section on the

(*) This represents to use only an absolute value for the bending moment at the central support, acting with opposite signs to each span (biaction).

segment "a"

H_{na}, V_{na}, M_{na}	= External actions at joint "n" on segment "a"
$\Theta^{(1)}_{kia}, \Theta^{(2)}_{kia}$	= Rotations at transversal extreme "k" of plate "i" and segment "a" corresponding respectively to the first and second transversal analysis.
$\Theta^{(1)}_{na}, \Theta^{(2)}_{na}$	= Item. rotation at joint "n" on segment "a"
$\mu^{(1)}_{kia}, \mu^{(2)}_{kia}$	= Bending moment at transversal extreme "k" of plate "i" and segment "a", corresponding to the first and second transversal analysis.
$\gamma^{(1)}_{kia}, \gamma^{(2)}_{kia}$	= Item. shear forces.
$\nu^{(1)}_{kia}, \nu^{(2)}_{kia}$	= Item. axial forces.
$\bar{\mu}^{(1)}_{kia}, \bar{\mu}^{(2)}_{kia}$	= Item. fixed-end moments.
$\bar{\nu}_{kia}, \bar{\gamma}_{kia}$	= Fixed-end axial and shear forces corresponding to the first transversal analysis.
$R^{(1)}_{na}, R^{(2)}_{na}$	= Support reaction at joint "n" on segment a corresponding to the first and second transversal analysis.
r_{kia}	= Stiffness coefficient.
g_{kia}	= Carry-over factor.
N_j	= Set of plates "i" wich extreme "j" coincides with the joint "n".
$\delta_{nn'}$	= Kronecker delta.
N_r, N_f	= Number of actual and ficticious imposed supports respectively.
Δ	= Segment length.
p_{ia}	= In-plane force at segment "a" of plate "i".
G_{1i}, G_{2i}	= Bending moments at longitudinal extremes 1 and 2 of the plate "i".
ML_{ia}	= Longitudinal bending moment at central cross-section on the segment "a" of the plate "i" without considering longitudinal compatibility.
QL_{ia}	= Item. shear force.
NL_{ia}	= Item. axial force.
q_{kia}	= Unit edge shearing stress at segment "a" of the plate "i".
T_{kia}	= Resultant edge shear force at segment "a" of the plate "i".
L_{na}	= Force per unit length of segment "a" along joint "n".
f_{nia}	= Longitudinal bending stress.

$\overline{ML}_{ia}, \overline{QL}_{ia}, \overline{NL}_{ia}$	=	Total longitudinal stress-resultants
E	=	Modulus of elasticity of the material.
A_{ia}	=	Cross-sectional area of plate "i" at segment "a".
Z_{mia}	=	Section modulus of the plate "i" at segment "a" corresponding to the edge "m"
t_{mia}	=	Thickness of the plate "i" at segment "a" corresponding to the edge "m".
α_{mia}	=	Taper angle at edge "m" of the plate "i" on segment "a".
h_{mia}	=	Distance between neutral axis and edge "m" of the plate "i" at segment "a".
I_{ia}	=	Moment of inertia of the plate "i" at segment "a".
Θ_{1i}, Θ_{2i}	=	End rotations at longitudinal extremes 1 and 2 of the plate "i".
u_{ia}	=	In-plane deflection of the plate "i" at segment "a".
ϕ_{ia}	=	Transversal angle at segment "a" between the plate "i" and the horizontal.
ω_{ia}	=	Transversal rotation of the plate "i" at segment "a" corresponding to the second transversal analysis.

$$\underline{S}_1 = \frac{1}{16} \times \begin{bmatrix} 5 & 4 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 8 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 8 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \cdots & -1 & 8 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 8 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -4 & 11 \end{bmatrix} \qquad \underline{S}_8 = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 1 & \cdots & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

$$\underline{S}_2 = \frac{1}{16} \times \begin{bmatrix} 11 & -4 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 8 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 8 & -1 & \cdots & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \cdots & 1 & 8 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 8 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 4 & 5 \end{bmatrix} \qquad \underline{S}_7 = \begin{bmatrix} 0 & \Delta & 2\Delta & \cdots & (A-1)\Delta \\ 0 & 0 & \Delta & \cdots & (A-2)\Delta \\ 0 & 0 & 0 & \cdots & (A-3)\Delta \\ \hline 0 & 0 & 0 & \cdots & 2\Delta \\ 0 & 0 & 0 & \cdots & \Delta \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\underline{S}_3 = \frac{\Delta}{384} \times \left[\begin{array}{cccccccc} -95 & -66 & 17 & 0 \cdots 0 & 0 & 0 & 0 \\ -7 & 46 & 9 & 0 \cdots 0 & 0 & 0 & 0 \\ 0 & -7 & 46 & 9 \cdots 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \cdots -7 & 46 & 9 & 0 \\ 0 & 0 & 0 & 0 \cdots 0 & -7 & 46 & 9 \\ 0 & 0 & 0 & 0 \cdots 0 & 33 & -130 & 337 \end{array}\right]$$

$$\underline{S}_4 = \frac{\Delta}{384} \times \left[\begin{array}{cccccccc} -337 & 130 & -33 & 0 \cdots 0 & 0 & 0 & 0 \\ -9 & -46 & 7 & 0 \cdots 0 & 0 & 0 & 0 \\ 0 & -9 & -46 & 7 \cdots 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \cdots -9 & -46 & 7 & 0 \\ 0 & 0 & 0 & 0 \cdots 0 & -9 & -46 & 7 \\ 0 & 0 & 0 & 0 \cdots 0 & -17 & 66 & 99 \end{array}\right]$$

$$\underline{S}_{c3} = \frac{\Delta^2}{384} \times \left[\begin{array}{cccccccc} -101 & -54 & 11 & 0 \cdots 0 & 0 & 0 & 0 \\ -5 & 42 & 11 & 0 \cdots 0 & 0 & 0 & 0 \\ 0 & -5 & 42 & 11 \cdots 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \cdots -5 & 42 & 11 & 0 \\ 0 & 0 & 0 & 0 \cdots 0 & -5 & 42 & 11 \\ 0 & 0 & 0 & 0 \cdots 0 & 43 & -150 & 347 \end{array}\right]$$

$$\underline{S}_{c4} = \frac{\Delta^2}{384} \times \left[\begin{array}{cccccccc} -347 & 150 & -43 & 0 \cdots 0 & 0 & 0 & 0 \\ -11 & -42 & 5 & 0 \cdots 0 & 0 & 0 & 0 \\ 0 & -11 & -42 & 5 \cdots 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \cdots -11 & -42 & 5 & 0 \\ 0 & 0 & 0 & 0 \cdots 0 & -11 & -42 & 5 \\ 0 & 0 & 0 & 0 \cdots 0 & -11 & 54 & 101 \end{array}\right]$$

$$\underline{S}_{c1} = \frac{\Delta}{24} \times \left[\begin{array}{cccccccc} 8 & 5 & -1 & 0 \cdots 0 & 0 & 0 & 0 \\ -1 & 11 & 2 & 0 \cdots 0 & 0 & 0 & 0 \\ 0 & -1 & 11 & 2 \cdots 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \cdots -1 & 11 & 2 & 0 \\ 0 & 0 & 0 & 0 \cdots 0 & -1 & 11 & 2 \\ 0 & 0 & 0 & 0 \cdots 0 & 2 & -7 & 17 \end{array}\right]$$

$$\underline{S}_{c2}=\frac{\Delta}{24}\times\left[\begin{array}{cccccccc}17 & -7 & 2 & 0 & \cdots & 0 & 0 & 0 & 0\\ 2 & 11 & -1 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & 2 & 11 & -1 & \cdots & 0 & 0 & 0 & 0\\ \hdashline 0 & 0 & 0 & 0 & \cdots & 2 & 11 & -1 & 0\\ 0 & 0 & 0 & 0 & \cdots & 0 & 2 & 11 & -1\\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 5 & 8\end{array}\right]$$

$$\underline{ML}_i^*=\left[\begin{array}{c}\frac{1}{2}(h_{1i1}\cdot\cos\alpha_{1i1}\cdot T^*_{1i0}+h_{2i1}\cdot\cos\alpha_{2i1}\cdot T^*_{2i0})\\ 0\\ \hdashline 0\end{array}\right]$$

$$\underline{NL}_i^*=\left[\begin{array}{c}\frac{1}{2}(\cos\alpha_{1i1}\cdot T^*_{1i0}-\cos\alpha_{2i1}\cdot T^*_{2i0})\\ 0\\ \hdashline 0\end{array}\right]$$

$$\underline{QL}_i^*=\left[\begin{array}{c}-\frac{1}{2}(\sin\alpha_{1i1}\cdot T^*_{1i0}+\sin\alpha_{2i1}\cdot T^*_{2i0})\\ 0\\ \hdashline 0\end{array}\right]$$

$$\underset{(j=1,2)}{\underline{D}_{ji}=\frac{1}{2}\times}\left[\begin{array}{cccccc}h_{ji1}\cdot\cos\alpha_{ji1} & 0 & \cdots & 0 & 0\\ h_{ji2}\cdot\cos\alpha_{ji2} & h_{ji2}\cdot\cos\alpha_{ji2} & \cdots & 0 & 0\\ \hdashline 0 & 0 & \cdots & h_{jiA}\cdot\cos\alpha_{jiA} & h_{jiA}\cdot\cos\alpha_{jiA}\end{array}\right]$$

$$\underline{D}_{ki} = \frac{1}{2} \times \begin{pmatrix} \cos\alpha_{ki1} & 0 & \cdots & 0 & 0 \\ \cos\alpha_{ki2} & \cos\alpha_{ki2} & \cdots & 0 & 0 \\ \hdashline 0 & 0 & \cdots & \cos\alpha_{kiA} & \cos\alpha_{kiA} \end{pmatrix} \quad (k=3,4)$$

$$\underline{D}_{ri} = \frac{1}{2} \times \begin{pmatrix} \sin\alpha_{ri1} & 0 & \cdots & 0 & 0 \\ \sin\alpha_{ri2} & \sin\alpha_{ri2} & \cdots & 0 & 0 \\ \hdashline 0 & 0 & \cdots & \sin\alpha_{riA} & \sin\alpha_{riA} \end{pmatrix} \quad (r=5,6)$$

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